Multipole expansion

An electric dipole consists of two equal and opposite charges ($\pm q$) separated by a distance d. Now, we have to calculate the approximate potentials at points far from the dipole.

Let r_- be the distance from $-q$ and r_+ the distance from $+q$.

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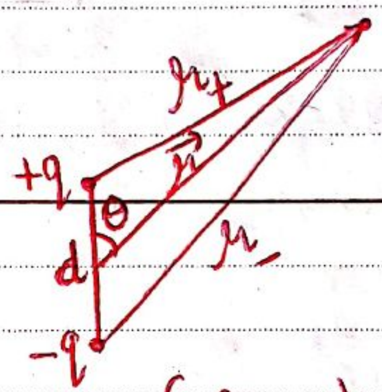

(fig. 1)

Then,

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_+} - \frac{q}{r_-}\right)$$

and (from the law of cosines)

$$r_{\pm} = r^2 + \left(\frac{d}{2}\right)^2 \mp rd\cos\theta$$

$$r_{\pm}^2 = r^2\left(1 \mp \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right)$$

In the regime $r \gg d$, the third term is negligible, and the binomial expansion gives,

$$\frac{1}{r_{\pm}} \cong \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-1/2}$$

$$\frac{1}{r_{\pm}} \cong \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)$$

Thus,

$$\frac{1}{r_+} - \frac{1}{r_-} \cong \frac{d}{r^2}\cos\theta$$

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and hence $$\boxed{V(\vec{r}) \cong \frac{1}{4\pi\epsilon_0}\cdot\frac{qd\cos\theta}{r^2}} \quad (1)$$

It is evident from eqⁿ (1), that the potential of a dipole goes like $\frac{1}{r^2}$ at large r; i.e. it falls off more rapidly than the potential of a point charge.

Similarly, If we put together a pair of equal and opposite dipoles to make a quadrupole, the potential goes like $\frac{1}{r^3}$; for back-to-back quadrupoles (an octopole) it goes like $\frac{1}{r^4}$; and so on. For monopole (point charge), potential goes like $\frac{1}{r}$.

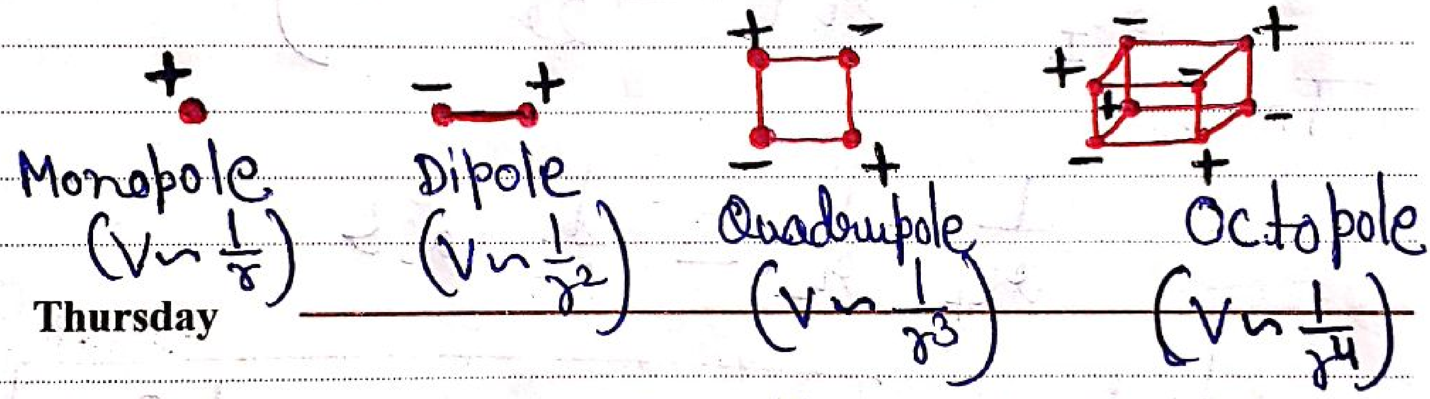

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Fig.1 — Potential of different poles

Now, we will develop a systematic expansion for the potential of an arbitrary localized charge distribution in power of $\frac{1}{r}$.

Fig. 2 defines the appropriate variables; the potential at $\vec{r}$ is given by,

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{1}{\mathscr{r}} \rho(\vec{r}') \cdot d\tau' \quad \text{——— (2)}$$

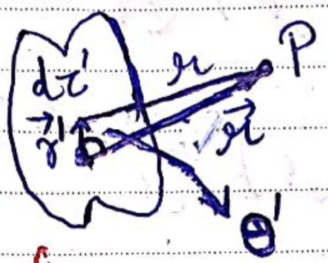

fig. 2

Using the law of cosines,

$$\mathscr{r}^2 = r^2 + (r')^2 - 2r.r' \cos\theta'$$

$$= r^2 \left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\theta'\right]$$

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or, $$\mathscr{r} = r\sqrt{1+\epsilon} \quad \text{——— (3)}$$

where, $$\epsilon = \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\theta'\right)$$

for points outside the charge distribution, ϵ is much less than 1, and this gives a binomial expansion:

$$\frac{1}{\mathscr{r}} = \frac{1}{r}(1+\epsilon)^{-1/2} = \frac{1}{r}\left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \dots\right)$$

Or, in terms of r, r' and θ':

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\theta'\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\theta'\right)^2 - \frac{5}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r} - 2\cos\theta'\right)^3 + \ldots\right]$$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)(\cos\theta') + \left(\frac{r'}{r}\right)^2 \frac{(3\cos^2\theta' - 1)}{2} + \left(\frac{r'}{r}\right)^3 \frac{(5\cos^3\theta' - 3\cos\theta')}{2} + \ldots\right] \quad (4)$$

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In the eq^n (4), the coefficients of the powers of $\left(\frac{r'}{r}\right)$ are Legendre polynomials. The result is

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta') \quad (5)$$

Where, θ' is the angle between $\vec{r}$ and $\vec{r}'$.

Substituting this back into eq^n (2), and noting that r is a constant, as far as the integration is concerned,

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int (r')^n P_n(\cos\theta') \rho(\vec{r'}) d\tau' \quad (6)$$

or,

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int \rho(\vec{r'})d\tau' + \frac{1}{r^2}\int r'\cos\theta' \rho(\vec{r'})d\tau' + \frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)\rho(\vec{r'})d\tau' + \cdots\right]} \quad (7)$$

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This is required result — the multipole expansion of V in powers of $\frac{1}{r}$. The first term ($n=0$) is the monopole contribution (it goes like $\frac{1}{r}$); the second ($n=1$) is the dipole (it goes like $\frac{1}{r^2}$); the third is quadrupole; the fourth octopole; and so on.